

THE ADDITIVE COMPLEMENTS OF PRIMES AND GOLDBACH'S PROBLEM

LI-XIA DAI AND HAO PAN

1. INTRODUCTION

First, we introduce the notion of additive complements. For a set $A \subseteq \mathbb{N} = \{0, 1, 2, 3, \dots\}$, we say a set $B \subseteq \mathbb{N}$ is an additive complement of A , provided that for every sufficiently large $n \in \mathbb{N}$, there exist $a \in A$ and $b \in B$ such that $n = a + b$, i.e., the sumset

$$A + B = \{a + b : a \in A, b \in B\}$$

contains all sufficiently large integers.

Furthermore, for the sets $A, B \subseteq \mathbb{N}$, if the sumset $A + B$ has lower density 1, i.e., almost all positive integers n can be represented as $n = a + b$ with $a \in A$ and $n \in B$, then we say B is an almost additive complement of A .

The additive properties of primes are always one of the most fascinating topics in number theory. Let \mathcal{P} denote the set of all primes. It is natural to ask what the additive complements of \mathcal{P} are. By the prime number theorem, we know that

$$\mathcal{P}(x) \sim \frac{x}{\log x},$$

where $A(x) = |A \cap [1, x]|$ for a set $A \subseteq \mathbb{N}$. So if A is an additive complement of \mathcal{P} , we must have $A(x) \gg \log x$. Unfortunately, no one knows whether there exists an additive complement A of \mathcal{P} satisfying $A(x) = O(\log x)$. However, in [2], using the probability method, Erdős showed that such additive complement exists, provided that we replace $\log x$ by $(\log x)^2$. That is,

there exists a set $A \subseteq \mathbb{N}$ with $A(x) = O((\log x)^2)$ such that the sumset $A + \mathcal{P}$ contains every sufficiently large integer.

In 1998, Ruzsa [8] improved the results of Wolke [12] and Kolountzakis[7], and showed that for every function $w(x)$ with $\lim_{x \rightarrow +\infty} w(x) = +\infty$, there exists an almost additive complement A of \mathcal{P} with $A(x) = O(w(x) \log x)$, i.e.,

there exists a set $A \subseteq \mathbb{N}$ with $A(x) = O(w(x) \log x)$ such that almost all positive integers can be represented as the sums of one element of A and a prime.

In 2001, Vu proved that \mathcal{P} has an additive complement A of order 2 with $A(x) = O(\log x)$, i.e.,

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there exists a set $A \subseteq \mathbb{N}$ with $A(x) = O(\log x)$ such that every sufficiently large integer n can be represented as $n = a_1 + a_2 + p$, where $a_1, a_2 \in A$ and p is a prime.

Clearly Vu's result implies Erdős' result, since $(A + A)(x) \ll (A(x))^2$.

Next, let us turn to the Goldbach problem. As early as 1937, using the circle method, Vinogradov [10] had solved the ternary Goldbach problem and showed that

every sufficiently large odd integer can be represented as the sum of three primes.

And subsequently, with a similar discussion, Estermann [3] proved the binary Goldbach problem is true for almost all positive even integers, i.e.,

almost all positive even integers can be represented as the sums of two primes.

In this note, we shall combine the results of Ruzsa and Vu with Goldbach's problem.

Theorem 1.1. *There exists a set $\mathcal{A} \subseteq \mathcal{P}$ with $\mathcal{A}(x) = O(\log x)$ such that every sufficiently large odd integer can be represented as $a_1 + a_2 + p$ where $a_1, a_2 \in \mathcal{A}$ and $p \in \mathcal{P}$.*

Theorem 1.2. *For every function $w(x)$ with $\lim_{x \rightarrow +\infty} w(x) = +\infty$, there exists a set $\mathcal{B} \subseteq \mathcal{P}$ with $\mathcal{B}(x) = O(w(x) \log x)$ such that almost all even positive integers can be represented as $b + p$ where $b \in \mathcal{B}$ and $p \in \mathcal{P}$.*

Note that Theorems 1.1 and 1.2 imply Vu's and Ruzsa's results respectively. If we set $A = \mathcal{A} \cup (\mathcal{A} + \{1\})$ and $B = \mathcal{B} \cup (\mathcal{B} + \{1\})$, then clearly the sumset $A + A + \mathcal{P}$ contains all sufficiently large integer and $B + \mathcal{P}$ has lower density 1.

The proofs of Theorems 1.1 and 1.2 will be given in the next sections.

2. PROOF OF THE THEOREM 1.1

The key of our proofs is the following lemma.

Lemma 2.1. *There exists a positive constant c_0 such that if x is sufficiently large, $x^{1-c_0} \leq M \leq x$ and $0 \leq y \leq x - M$, then for all even integers n with $x \leq n \leq x + M$, except for $O(M(\log x)^{-2})$ exceptional values, we always have*

$$\sum_{\substack{n=p_1+p_2 \\ y \leq p_1 \leq y+M \\ x-y-M \leq p_2 \leq x-y+M}} 1 \gg C(n) \frac{M}{(\log x)^2} \quad (2.1)$$

where

$$C(n) = \prod_{p \nmid n} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|n} \left(1 + \frac{1}{p-1}\right) \quad (2.2)$$

and the implied constant in (2.1) only depends on c_0 .

Proof. This lemma can be proved by the method of Jia in [6], although he only discussed the case $y = x/2$. In fact, Jia proved that Lemma 2.1 holds whenever $c_0 > 7/12$. \square

Now suppose that n is a sufficiently integer. For each $x \in \mathcal{P}$, we choose x to be in \mathcal{A} with the probability

$$\varrho_x = \frac{\mathbf{c} \log x}{x},$$

where $\mathbf{c} > 0$ is a constant to be chosen later. Then using the law of large numbers, one can easily show that almost surely $\mathcal{A}(n) = O(\log n)$ for every n , since

$$\sum_{\substack{p \leq n \\ p \text{ is prime}}} \frac{\log p}{p} \sim \log n.$$

Let t_x be the binary random variable representing the choice of x , i.e., $t_x = 1$ with probability ϱ_x and 0 with probability $1 - \varrho_x$. Consider

$$Y_n = \sum_{\substack{p < n \\ p \text{ is prime}}} \sum_{\substack{i+j=n-p \\ i, j \text{ are prime}}} t_i t_j.$$

We need to prove that there exists $n_0 > 0$ satisfying the probability

$$\mathbb{P}(Y_n > 0 \text{ for every } n \geq n_0) \geq 1/2.$$

Choose $0 < \epsilon < c_0/2$ and let $M = n^{1-2\epsilon}$, where c_0 is the one appearing in Lemma 2.1. In the remainder of this section, the implied constants of $O(\cdot)$, \ll and \gg will only depend on ϵ . Let

$$Y_n^* = \sum_{\substack{p \leq n - n^{1-\epsilon} \\ p \text{ is prime}}} \sum_{\substack{i+j=n-p \\ i, j \geq M \\ i, j \text{ are prime}}} t_i t_j.$$

Clearly $\mathbb{P}(Y_n > 0) \geq \mathbb{P}(Y_n^* > 0)$ for every n . The following lemma is due to Janson [5].

Lemma 2.2. *Let z_1, \dots, z_m be independent indicator random variables and $Y = \sum_{\alpha} I_{\alpha}$ where each I_{α} is the product of few z_j 's. Define $\alpha \sim \beta$ if there is some z_j which is in both I_{α} and I_{β} . Let $\Delta = \sum_{\alpha \sim \beta} \mathbb{E}(I_{\alpha} I_{\beta})$. For any Y and any positive number ϵ we have*

$$\mathbb{P}(Y \leq (1 - \epsilon)\mathbb{E}(Y)) \leq \exp\left(-\frac{(\epsilon\mathbb{E}(Y))^2}{2(\mathbb{E}(Y) + \Delta)}\right).$$

Let

$$X = \{(i, j) : i, j \geq M, i + j \geq n^{1-\epsilon} \text{ and } i, j, n - i - j \text{ are all prime}\}.$$

Clearly,

$$Y_n^* = \sum_{(i, j) \in X} t_i t_j = \sum_{\alpha \in X} I_{\alpha},$$

where $I_{\alpha} = t_i t_j$ for $\alpha = (i, j)$. In view of Lemma 2.2 we only need to estimate

$$\mathbb{E}(Y_n^*) = \mathbb{E}\left(\sum_{\alpha \in X} I_{\alpha}\right) \quad \text{and} \quad \Delta = \mathbb{E}\left(\sum_{\substack{\alpha, \beta \in X \\ \alpha \sim \beta}} I_{\alpha} I_{\beta}\right).$$

By Lemma 2.1, we have

$$\begin{aligned}
\mathbb{E}\left(\sum_{\alpha \in X} I_\alpha\right) &= \mathbb{E}\left(\sum_{\substack{p \leq n-n^{1-\epsilon} \\ p \text{ is prime}}} \sum_{\substack{i+j=n-p \\ i, j \geq M \\ i, j \text{ are prime}}} t_i t_j\right) \geq \mathbb{E}\left(\sum_{\substack{0 \leq t \leq n^{1-\epsilon}/M \\ tM \leq n-n^{1-\epsilon}-p < (t+1)M \\ p \text{ is prime}}} \sum_{\substack{1 \leq s \leq (n-p)/M-2 \\ i+j=n-p \\ sM \leq i < (s+1)M \\ i, j \text{ are prime}}} t_i t_j\right) \\
&\geq \mathbb{E}\left(\sum_{\substack{0 \leq t \leq n^{1-\epsilon}/M \\ 1 \leq s \leq n^{1-\epsilon}/M+t-2}} \sum_{\substack{tM \leq n-n^{1-\epsilon}-p < (t+1)M \\ p \text{ is prime}}} \sum_{\substack{i+j=n-p \\ sM \leq i < (s+1)M \\ i, j \text{ are prime}}} t_i t_j\right) \\
&\gg \sum_{\substack{0 \leq t \leq n^{1-\epsilon}/M \\ 1 \leq s \leq n^{1-\epsilon}/M+t-2}} \frac{M}{\log M} \cdot \frac{M}{(\log M)^2} \cdot \frac{\mathbf{c}^2 (\log M)^2}{sM \cdot (n^{1-\epsilon} + (t+1)M - sM)}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
&\sum_{\substack{0 \leq t \leq n^{1-\epsilon}/M \\ 1 \leq s \leq n^{1-\epsilon}/M+t-2}} \frac{1}{sM \cdot (n^{1-\epsilon} + (t+1)M - sM)} \\
&\gg \int_1^{n^{1-\epsilon}/M} \left(\int_1^{n^{1-\epsilon}/M+t-2} \frac{1}{sM(n^{1-\epsilon} + tM - sM)} ds \right) dt \\
&\gg \int_1^{n^{1-\epsilon}/M} \frac{1}{n^{1-\epsilon} + tM} \left(\int_1^{n^{1-\epsilon}/M+t-2} \left(\frac{1}{sM} + \frac{1}{n^{1-\epsilon} + tM - sM} \right) ds \right) dt \\
&\gg \int_1^{n^{1-\epsilon}/M} \frac{\log(n^{1-\epsilon} + tM)}{M(n^{1-\epsilon} + tM)} dt \gg \frac{(\log(2n^{1-\epsilon}))^2}{M^2} \gg \frac{(\log n)^2}{M^2}.
\end{aligned}$$

Thus we get that

$$\mathbb{E}(Y_n^*) \gg \mathbf{c}^2 \log n.$$

Now we turn to Δ .

$$\begin{aligned}
\mathbb{E}\left(\sum_{\substack{\alpha, \beta \in X \\ \alpha \sim \beta}} I_\alpha I_\beta\right) &= \mathbb{E}\left(\sum_{\substack{i \geq M \\ i \text{ is prime}}} \sum_{\substack{p_1+j_1=p_2+j_2=n-i \\ p_1, p_2 \leq n-n^{1-\epsilon} \\ j_1, j_2 \geq M \\ p_1, p_2, j_1, j_2 \text{ are prime}}} (t_i t_{j_1})(t_i t_{j_2})\right) \\
&\leq \sum_{\substack{i \geq M \\ i \text{ is prime}}} \frac{\mathbf{c} \log i}{i} \sum_{\substack{p_1+j_1=p_2+j_2=n-i \\ p_1, p_2 \leq n-n^{1-\epsilon} \\ j_1, j_2 \geq M \\ p_1, p_2, j_1, j_2 \text{ are prime}}} \frac{\mathbf{c} \log j_1}{j_1} \cdot \frac{\mathbf{c} \log j_2}{j_2} \\
&\leq \sum_{\substack{M \leq i \leq n-n^{1-\epsilon} \\ i \text{ is prime}}} \frac{\mathbf{c} \log n}{i} \left(\sum_{\substack{p+j=n-i \\ j \geq M \\ p, j \text{ are prime}}} \frac{\mathbf{c} \log n}{j} \right)^2.
\end{aligned}$$

For a set U of positive integers, by the partial summation, we have

$$\sum_{\substack{M \leq j \leq x \\ j \in U}} \frac{1}{j} \ll \frac{U(x)}{x} - \frac{U(M)}{M} + \sum_{M < y \leq x} \frac{U(y)}{y^2}.$$

Note that by the sieve method, we have

$$|M \leq j \leq y : \text{both } j \text{ and } m - j \text{ are primes}| \ll \frac{C(m)(y - M)}{(\log(y - M))^2} \ll \frac{C(m)y}{(\log y)^2}$$

since $y/(\log y)^2$ is increasing for $y > e$, where $C(\cdot)$ is the one appearing in Lemma 2.1. Hence

$$\begin{aligned} \sum_{\substack{p+j=n-i \\ j \geq M \\ p, j \text{ are prime}}} \frac{1}{j} &\ll \frac{1}{(n-i)} \cdot \frac{C(n-i) \cdot (n-i)}{(\log(n-i))^2} + \frac{1}{M} + \sum_{M+e < y \leq n-i} \frac{C(n-i)}{y(\log y)^2} \\ &\ll \frac{C(n-i)}{\log M} \ll \frac{C(n-i)}{\log n}. \end{aligned}$$

Thus,

$$\mathbb{E} \left(\sum_{\alpha \sim \beta} I_\alpha I_\beta \right) \ll \mathbf{c}^3 \log n \sum_{\substack{M \leq i \leq n-n^{1-\epsilon} \\ i \text{ is prime}}} \frac{C(n-i)^2}{i}.$$

Note that

$$C(n-i)^2 \ll \prod_{p|n-i} \left(1 + \frac{1}{p-1} \right)^2 \ll \prod_{p|n-i} \left(1 + \frac{1}{p} \right)^2 \ll \prod_{p|n-i} \left(1 + \frac{2}{p} \right) = \sum_{\substack{d|n-i \\ d \text{ is square-free}}} \frac{2^{\omega(d)}}{d},$$

where $\omega(d)$ denotes the number of the distinct prime factors of d . Hence,

$$\sum_{\substack{M \leq i \leq n-n^{1-\epsilon} \\ i \text{ is prime}}} \frac{C(n-i)^2}{i} \ll \sum_{\substack{d < n-n^{1-\epsilon}-M \\ d \text{ is square-free}}} \frac{2^{\omega(d)}}{d} \sum_{\substack{M \leq i \leq n-n^{1-\epsilon} \\ i \text{ is prime} \\ i \equiv n \pmod{d}}} \frac{1}{i}$$

By the Brun-Titchmarsh theorem, we know

$$|\{M \leq i \leq y : i \text{ is prime and } i \equiv n \pmod{d}\}| \ll \frac{y - M}{\phi(d) \log((y - M)/d)},$$

provided that $y - M \geq 1.1d$. By the partial summation, for $d \leq (n - n^{1-\epsilon} - M)/1.1$,

$$\begin{aligned} &\sum_{\substack{M \leq i \leq n-n^{1-\epsilon} \\ i \text{ is prime} \\ i \equiv n \pmod{d}}} \frac{1}{i} \\ &\ll \frac{1}{n - n^{1-\epsilon}} \cdot \frac{n}{\phi(d) \log(n/d)} + \frac{1}{M} + \sum_{M < y < M+1.1d} \frac{2}{y^2} + \sum_{M+1.1d \leq y \leq n-n^{1-\epsilon}} \frac{1}{\phi(d)y \log(y/d)} \\ &\ll \frac{1}{\phi(d)} + \frac{1}{M} + \frac{\log \log(n/d) - \log \log(M/d)}{\phi(d)}. \end{aligned}$$

If $d \leq \sqrt{M}$, then

$$\log \log(n/d) - \log \log(M/d) \leq \log \log n - \log \log \sqrt{M} \ll 1.$$

If $\sqrt{M} < d \leq (n - n^{1-\epsilon} - M)/1.1$, then

$$\frac{\log \log(n/d) - \log \log(M/d)}{\phi(d)} \ll \frac{\log \log n}{d^{1-\epsilon}} \ll \frac{1}{M^{1/2-\epsilon}}.$$

Thus we have

$$\begin{aligned} & \sum_{\substack{d \leq (n - n^{1-\epsilon} - M)/1.1 \\ d \text{ is square-free}}} \frac{2^{\omega(d)}}{d} \sum_{\substack{M \leq i \leq n - n^{1-\epsilon} \\ i \text{ is prime} \\ i \equiv n \pmod{d}}} \frac{1}{i} \ll \sum_{d < n} \frac{2^{\omega(d)}}{d} \left(\frac{1}{M^{1/2-\epsilon}} + \frac{1}{\phi(d)} \right) \\ & \ll \frac{1}{M^{1/2-\epsilon}} \sum_{d < n} \frac{1}{d^{1-\epsilon}} + \sum_{d < n} \frac{1}{d^{2-\epsilon}} \ll \frac{n^\epsilon}{M^{1/2-\epsilon}} + O(1) = O(1). \end{aligned} \quad (2.3)$$

On the other hand, we have

$$\begin{aligned} & \sum_{\substack{(n - n^{1-\epsilon} - M)/1.1 < d \leq n - n^{1-\epsilon} - M \\ d \text{ is square-free}}} \frac{2^{\omega(d)}}{d} \sum_{\substack{M \leq i \leq n - n^{1-\epsilon} \\ i \text{ is prime} \\ i \equiv n \pmod{d}}} \frac{1}{i} \\ & \ll \sum_{\substack{(n - n^{1-\epsilon} - M)/1.1 < d \leq n - n^{1-\epsilon} - M \\ d \text{ is square-free}}} \frac{2^{\omega(d)}}{n} \sum_{\substack{M \leq i \leq n - n^{1-\epsilon} \\ i \text{ is prime} \\ i = n - d}} 1 \leq \frac{1}{n} \sum_{\substack{i < n \\ i \text{ is prime}}} \tau(n - i), \end{aligned}$$

where τ be the divisor function and $\tau(d) = 2^{\omega(d)}$ if d is square-free. Now

$$\begin{aligned} \sum_{\substack{i < n \\ i \text{ is prime}}} \tau(n - i) &= \sum_{\substack{i < n \\ i \text{ is prime}}} \sum_{k | n - i} 1 \leq 2 \sum_{\substack{i < n \\ i \text{ is prime}}} \sum_{\substack{k \leq \sqrt{n - i} \\ k | n - i}} 1 \\ &\leq \sum_{k \leq \sqrt{n}} \sum_{\substack{i < n \\ i \text{ is prime} \\ i \equiv n \pmod{k}}} 1 \ll \sum_{k \leq \sqrt{n}} \frac{n}{\phi(k) \log n}. \end{aligned}$$

We know that (cf. [9])

$$\sum_{k \leq x} \frac{1}{\phi(k)} \ll \log x.$$

It follows that

$$\sum_{\substack{(n - n^{1-\epsilon} - M)/1.1 < d \leq n - n^{1-\epsilon} - M \\ d \text{ is square-free}}} \frac{2^{\omega(d)}}{d} \sum_{\substack{M \leq i \leq n - n^{1-\epsilon} \\ i \text{ is prime} \\ i \equiv n \pmod{d}}} \frac{1}{i} \ll 1. \quad (2.4)$$

Thus

$$\Delta \ll \mathbf{c}^3 \log n.$$

Now

$$\frac{\mathbb{E}(Y_n^*)^2}{\mathbb{E}(Y_n^*) + \Delta} = \frac{\mathbb{E}(Y_n^*)}{1 + \frac{\Delta}{\mathbb{E}(Y_n^*)}} \gg \frac{\mathbf{c}^2 \log n}{1 + \frac{\mathbf{c}^3 \log n}{\mathbf{c}^2 \log n}} = \frac{\mathbf{c}^2}{1 + \mathbf{c}} \log n.$$

So we may choose sufficiently large \mathbf{c} such that

$$\frac{\mathbb{E}(Y_n^*)^2}{\mathbb{E}(Y_n^*) + \Delta} \geq 100 \log n.$$

In view of Lemma 2.2, we have

$$\mathbb{P}(Y_n^* = 0) \leq \mathbb{P}(Y_n^* \leq 0.5\mathbb{E}(Y_n^*)) \leq \exp\left(-\frac{(0.5\mathbb{E}(Y_n^*))^2}{2(\mathbb{E}(Y_n^*) + \Delta)}\right) \leq \exp(-2 \log n) = \frac{1}{n^2}.$$

Thus choosing a sufficiently large n_0 , we have

$$\mathbb{P}(Y_n^* = 0 \text{ for some } n \geq n_0) \leq \sum_{n \geq n_0} \mathbb{P}(Y_n^* = 0) \leq \sum_{n \geq n_0} \frac{1}{n^2} \leq \frac{1}{2}.$$

This completes the proof of Theorem 1.1.

3. PROOF OF THE THEOREM 1.2

Let c_0 be the constant appearing in Lemma 2.1 and c_1 be an another fixed constant with $c_0 < c_1 < 1$.

Lemma 3.1. *For every $0 < \epsilon < 1$, there is a $K = K(\epsilon)$ and an $N_0 = N_0(\epsilon)$ such that for $N > N_0$ we can always find a set*

$$B \subset [N^{c_0}, 2N^{c_0}]$$

of primes such that

$$|B| \leq K \log N \tag{3.1}$$

and the set $S = \mathcal{P} + B$ satisfies

$$S(x) \geq (1 - \epsilon)x \text{ for all } N^{c_1} \leq x \leq N. \tag{3.2}$$

Proof. Assume that N is sufficiently large. Suppose that C^* is the implied constant in (2.1). Let

$$K = \max\left\{\frac{10}{c_0 C^*} \log \frac{16}{\epsilon^2}, 20\right\}, \quad M = N^{c_0}, \quad I = [M, 2M] \cap \mathcal{P}, \quad L = |I| = (1 + o(1)) \frac{M}{\log M}.$$

Let B be a random subset of I such that each $x \in I$ is included into B independently with a probability

$$\varrho = \frac{K \log N}{2L}.$$

And let t_x be the binary random variable representing the choice of x . Clearly

$$\mathbb{E}(|B|) = \mathbb{E}\left(\sum_{x \in I} t_x\right) = \varrho L = \frac{K}{2} \log N.$$

Now it suffices to show that

$$\mathbb{P}((3.2) \text{ holds}) > \mathbb{P}(|B| > K \log N). \tag{3.3}$$

Apparently,

$$\begin{aligned}\mathbb{E}(e^{|B|}) &= \mathbb{E}\left(\exp\left(\sum_{x \in I} t_x\right)\right) = \prod_{x \in I} \mathbb{E}(e^{t_x}) = ((1 - \varrho) \cdot e^0 + \varrho \cdot e^1)^L \\ &= (1 + \varrho(e - 1))^L \leq \exp(\varrho(e - 1)L) < \exp\left(\frac{K(e - 1)}{1.9} \log N\right).\end{aligned}$$

Thus from Markov's inequality we get

$$\begin{aligned}\mathbb{P}(|B| > K \log N) &= \mathbb{P}(e^{|B|} > e^{K \log N}) \leq \frac{\mathbb{E}(e^{|B|})}{e^{K \log N}} \\ &\leq \exp\left(\frac{K(e - 1)}{1.9} \log N - K \log N\right) = N^{\frac{K(e - 2.9)}{1.9}} < \frac{1}{N}.\end{aligned}$$

Below we only need to show that $\mathbb{P}((3.2) \text{ holds}) \geq N^{-1}$. Let $\eta = 1 + \epsilon/2$ and let $x_j = N/\eta^j$ for $0 \leq j \leq (1 - c_1) \log N / \log \eta + 1$. Clearly, for $x_j \leq x \leq x_{j-1}$, $S(x_j) > (1 - \epsilon/2)x_j$ implies that $S(x) > (1 - \epsilon)x$. So it suffices to show that

$$\mathbb{P}(S(x_j) > (1 - \epsilon/2)x_j \text{ for all } j) > \frac{1}{N}.$$

Let

$$T(x) = x - S(x).$$

Clearly $S(x_j) > (1 - \epsilon/2)x_j$ is equivalent to $T(x_j) < (\epsilon/2)x_j$.

Let

$$z(n) = \sum_{\substack{n=p_1+p_2 \\ p_1 \in I, p_2 \in \mathcal{P}}} 1$$

and

$$Q_j = \left\{M \leq n \leq x_j : z(n) < \frac{C^* C(n) M}{3 \log^2 n}\right\}$$

By Lemma 2.1, we have

$$|Q_j| \leq \sum_{1 \leq k \leq x_j/M} |Q_j \cap [kM, (k+1)M]| = \sum_{1 \leq k \leq x_j/M} O\left(\frac{M}{(\log(kM))^2}\right) = O\left(\frac{x_j}{(\log M)^2}\right).$$

For a given $n \subseteq [2M, x_j] \setminus Q_j$, we have $n \notin S$ only if none of those $z(n)$ primes

$$\{p_1 \in I : n - p_1 \text{ is prime}\}$$

is in B . The probability of this event is

$$\mathbb{P}(n \notin S) = (1 - \varrho)^{z(n)} \leq \exp(-\varrho z(n)).$$

We have

$$\varrho z(n) \geq \frac{K \log N}{2L} \cdot C^* C(n) \frac{M}{3 \log^2 n} \geq \frac{C^* C(n) K M}{6L \log N} \geq \frac{C^* c_0 K}{10},$$

since

$$C(n) \geq \prod_{p \geq 3 \text{ is prime}} \left(1 - \frac{1}{(p-1)^2}\right) > 0.6.$$

Hence

$$\mathbb{P}(n \notin S) \leq \exp(-0.1c_0C^*K). \quad (3.4)$$

Thus the expectation of $T(x_j)$ satisfies

$$\mathbb{E}(T(x_j)) \leq \exp(-0.1c_0C^*K)x_j + 2M + O(x_j(\log M)^{-2}) \leq 2\exp(-0.1c_0C^*K)x_j, \quad (3.5)$$

by noting that $x_j \geq N^{c_1}$. From Markov's inequality, we can infer

$$\mathbb{P}(T(x_j) \geq (\epsilon/2)x_j) \leq \frac{\mathbb{E}(T(x_j))}{(\epsilon/2)x_j} < \frac{4}{\epsilon} \exp(-0.1c_0C^*K) \leq 1 - \frac{1}{\eta}, \quad (3.6)$$

since

$$K \geq \frac{10}{c_0C^*} \log \frac{4}{\epsilon(1 - \eta^{-1})}. \quad (3.7)$$

So letting $J = \lfloor (1 - c_1) \log N / \log \eta + 1 \rfloor$, we have

$$\mathbb{P}(S(x_j) > (1 - \epsilon/2)x_j \text{ for all } j) \geq \prod_{j=0}^J \mathbb{P}(S(x_j) > (1 - \epsilon/2)x_j) \geq \eta^{-J-1} \geq \frac{1}{\eta^2 N^{1-c_1}} \geq \frac{1}{N}, \quad (3.8)$$

where $\lfloor \alpha \rfloor = \max\{z \in \mathbb{Z} : z \leq \alpha\}$. Thus the proof of Lemma 3.1 is complete. \square

Lemma 3.2. *For every $\epsilon > 0$, let $K = K(\epsilon)$ and $N_0 = N_0(\epsilon)$ be the ones defined in Lemma 3.1. Then there exists a set $B \subseteq \mathcal{P}$ such that for every $x > N_0$, we have*

$$B(x) \leq \frac{2K}{c_0c_1(1 - c_1)} \log x$$

and the sumset $S = \mathcal{P} + B$ satisfies

$$S(x) \geq (1 - \epsilon)x.$$

Proof. Define

$$N_{i+1} = \lfloor N_i^{1/c_1} \rfloor + 1$$

for $i \geq 0$. Applying Lemma 3.1 to each $N = N_i$, we get a set

$$B_i \subset [N_i^{c_0}, 2N_i^{c_0}]$$

with

$$|B_i| \leq K \log N_i$$

satisfying that

$$S_i(x) \geq (1 - \epsilon)x$$

for any $N_i^{c_1} \leq x \leq N_i$, where the set $S_i = \mathcal{P} + B_i$. We put

$$B = \bigcup B_i.$$

Suppose that $x \in [N_i^{c_1}, N_i]$. Clearly, for the set $S = \mathcal{P} + B$, we have

$$S(x) \geq S_i(x) \geq (1 - \epsilon)x.$$

Let

$$k = \min\{i : N_i^{c_0} > x \text{ and } N_{i-1}^{c_0} \leq x\}.$$

Note that

$$\log N_{i-1} \leq c_1 \log N_i.$$

Since $\log N_i$ grows exponentially, we get

$$B(x) \leq \sum_{0 \leq i \leq k} |B_i| \leq K(\log N_0 + \log N_1 + \cdots + \log N_k) \leq K \log N_k \sum_{j=0}^k c_1^j \leq \frac{2K \log(x^{1/c_0 c_1})}{1 - c_1}.$$

□

Next we are ready to prove Theorem 1.2.

Proof of the Theorem 1.2. Let

$$\epsilon_i = \frac{1}{i+1}.$$

Let B_i be the set satisfying the requirements of Lemma 3.2 for $\epsilon = \epsilon_i$. Let $N_1 = N_0(\epsilon_1)$. Since $w(x) \rightarrow \infty$ as $x \rightarrow \infty$, for every $i \geq 2$, let $N_i \geq \max\{N_0(\epsilon_i), e^{2N_{i-1}/\epsilon_i}\}$ be an integer such that

$$\frac{2}{c_0 c_1 (1 - c_1)} \sum_{j=1}^{i+1} K(\epsilon_j) \leq w(x)$$

for every $x \geq N_i$. Let

$$\mathcal{B} = \bigcup_{i=1}^{\infty} (B_i \cap [N_{i-1}, N_{i+1}]).$$

Note that for any $N_i < x \leq N_{i+1}$, we have

$$\begin{aligned} & |\{n \leq x : n = p + b, p \in \mathcal{P}, b \in \mathcal{B}\}| \\ & \geq |\{n \leq x : n = p + b, p \in \mathcal{P}, b \in B_i \cap [N_{i-1}, N_{i+1}]\}| \\ & > |\{n \leq x : n = p + b, p \in \mathcal{P}, b \in B_i\}| - N_{i-1} \cdot \mathcal{P}(x) \\ & \geq (1 - \epsilon_i)x - N_{i-1} \cdot \frac{2x}{\log x} \geq (1 - 2\epsilon_i)x. \end{aligned}$$

So the sumset $\mathcal{P} + \mathcal{B}$ has lower density 1. Also, for every $N_i < x \leq N_{i+1}$, we have

$$\mathcal{B}(x) \leq \sum_{j=1}^{i+1} |B_j \cap [1, x]| \leq \left(\frac{2}{c_0 c_1 (1 - c_1)} \sum_{j=1}^{i+1} K(\epsilon_j) \right) \log x \leq w(x) \log x.$$

□

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E-mail address: lilidainjnu@163.com

SCHOOL OF MATHEMATICAL SCIENCES, NANJING NORMAL UNIVERSITY, NANJING 210046, PEOPLE'S REPUBLIC OF CHINA

E-mail address: haopan79@yahoo.com.cn

DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, PEOPLE'S REPUBLIC OF CHINA